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SUMMARY

1. PURPOSE. To provide security and policy review on the document at Tab 1 prior to release to the public.

2. BACKGROUND.
Authors: Julie A. Barnes (Western Carolina University), Thomas L. Koehler (DFEG), Beth E. Schaubroeck (DFMS)

Title: Wind and Travel: How Meteorologists use Calculus to Predict Flight Times

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3. DISCUSSION. N/A

4. VIEWS OF OTHERS. N/A

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1 Tab
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Wind and Travel: How Meteorologists use Calculus to Predict Flight Times

Julie Barnes
Tom Koehler
Beth Schaubroeck

Suppose you want to visit New York City from San Francisco. According to Google Maps, this trip would take 47 hours to drive, each way! More likely, you would decide to save time and fly. The eastbound flight times from San Francisco to New York City are approximately 5.5 hours and the return flights are roughly 6.5 hours. Why is the eastbound flight so much shorter than the westbound? It is, after all, the same distance.

The explanation is probably not a surprise—it's the wind. But have you ever asked why the winds blow from west to east across the U.S.? And what do wind patterns have to do with multivariable calculus?

Upper Level Charts

Before we investigate the calculus behind wind patterns, we need to understand one of the standard tools used in aviation meteorology: upper level charts. Figure 1 shows an upper level chart of elevation for a fixed pressure of 250 mb (millibars are a measure of pressure) from April 15, 2012. You probably have seen maps like this while looking up weather information, but you have also seen diagrams resembling this in multivariable calculus.

[Figure 1 about here] Figure 1: Upper level chart of altitude of the 250 mb surface in the upper atmosphere

In multivariable calculus, we analyze functions by looking at level curves. For example, in figure 2, we see a topographical map from Palmer Lake, Colorado. The brown elevation contours indicate points that are at the same elevation; they are level curves for the elevation function. If you were to hike along one of those contours, the trail would be horizontal and easy to walk. If you were to walk perpendicular to a contour toward a contour representing a higher altitude, like from the Lower Reservoir toward the top of Sundance Mountain, you would be hiking straight uphill. In multivariable calculus we learn that the direction of steepest ascent is given by the gradient of the elevation function. The gradient vector is orthogonal to the level curves and its magnitude is the maximum rate of increase. The contours are very close together between the Lower Reservoir and the top of Sundance Mountain, meaning that the elevation changes quickly there and the slope of the land is very steep; this corresponds to a large gradient vector. Near the highway, indicated by the red and white line, the height contours are far apart, indicating that the surface is relatively level; the gradient is small.

[Figure 2 about here] Figure 2: A segment of the USGS quad sheet for Palmer Lake, CO

Twice each day, at hundreds of sites around the world, technicians launch weather balloons carrying a light, compact instrument package called a radiosonde (see figure 3). As the balloons rise the equipment measures pressure, temperature, humidity, and location. From the weather balloon data we obtain maps showing height and temperature contours at standard pressure levels throughout the atmosphere. For example, figure 1 shows the elevations where the pressure is 250 mb. This is the reverse of what we might expect; the map shows the altitude at a constant pressure, instead of the pressure at a constant altitude. The maps show altitude differences using height contours just like the topographical map in figure 2 does. That is, figure 1 shows the level curves of the function h, where $h(x,y)$ is the altitude of the 250 mb pressure surface over the point (x,y). The spacing between the contours is 120 m, or just under 400 ft. The lowest altitudes are in Canada while the highest altitudes are over Florida and the Gulf of Mexico. The steepest slopes on this surface range from northern Mexico to western Kansas where the contour lines are closest together.

[Figure 3 about here] Figure 3: Radiosonde balloon launch by the NWS office in Albuquerque, NM

Most long-distance airline flights in the U.S. have cruising altitudes of between 30,000 and 40,000 feet above sea level; because the 250 mb surface has an average height of 34,000 ft, it is a good representation of cruising altitude.

Wind

What meteorologists call wind is simply the velocity of the air, and we can use calculus to analyze this velocity. There are two main forces that describe the motion of the air: the pressure gradient force and the Coriolis effect. We will compute the wind, called the *geostrophic wind*, generated by these two forces. To do so we need a local coordinate system. Meteorologists use an *x*-axis pointing east, a *y*-axis pointing north, and a *z*-axis pointing upward, with zero being at sea level.

Wind is the velocity of the air, and the derivative of velocity is acceleration. Newton's second law of motion states that force is equal to mass times acceleration. Consider a parcel of air. The term "air parcel" has many definitions; here we define it to be one cubic meter of air. The mass of an air parcel equals the density, ρ, when it is measured per cubic meter. In this discussion, we refer to force per unit mass for the cubic meter parcel which has units of acceleration (m/s^2).

The pressure gradient Air wants to move from higher pressure locations to lower pressure locations. The gradient of the air pressure *p*, ∇p, called the *pressure gradient,* gives the direction that the pressure increases most rapidly. However, because we are only considering horizontal wind, we compute the gradient assuming a constant *z*, $-\nabla p(x,y,z_0) = -(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath})$. < edit equation>

The acceleration of a parcel due to the pressure gradient force then points in the opposite direction of the gradient and is equal to $-\frac{1}{\rho}(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath})$. Therefore, the horizontal pressure gradient force is perpendicular to the contour lines on the constant pressure chart, but directed toward lower pressure. It turns out that the horizontal gradient of pressure is directly proportional to the changes of height, *h,* of the constant pressure surface, and the constant of proportionality is precisely \rho g where *g* is the gravitational constant. (For the technical details, the interested reader can look to H. B. Bluestein's text, *Synoptic-Dynamic Meteorology in Midlatitudes: Principles of Kinematics and Dynamics, Vol. 1.*) This implies that $-\frac{1}{\rho}(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath}) = -g\nabla h$; now we have an expression for the acceleration in terms of *h.*

The Coriolis effect. The *Coriolis effect* produces an apparent force on objects on the Earth; it arises because our coordinate system is rotating with the Earth. Think of the Earth as a carousel. Suppose two children are riding horses directly opposite each other on the carnival ride, and one child tosses a ball to the other. Viewed from a stationary point above it, the ball travels in a straight path. However, to the children, the ball appears to curve, as if a horizontal force is acting on it; for their frame of reference is rotating with the carousel.

The angular velocity at a point on earth is a vector $\vec{\Omega}$ parallel to the axis of rotation; its magnitude is the rotational velocity of the Earth, Ω (2π radians per day), and its direction is determined by the right-hand rule (see figure 4). We use trigonometry to show that $\vec{\Omega} = 0\hat{\imath} + \Omega\cos(\phi)\hat{\jmath} + \Omega\sin(\phi)\hat{k}$, where ϕ is the latitude. Let $\vec{U}$ be the wind; that is, the velocity of the air. The Coriolis effect imparts to the wind an acceleration of $-2\vec{\Omega}\times\vec{U}$. (Again, see Bluestein's text for full details.)

[Figure 4 about here] Figure 4: Diagram of the vector from the Coriolis force. This diagram is not to scale.

The geostrophic wind approximation. At high altitudes, meteorologists typically assume that there is no net acceleration of the air. This is reasonable because any changes in velocity are small compared to the magnitude of the velocity; thus the acceleration is insignificant. By Newton's second law, the acceleration of an air parcel is directly proportional to, and in the same direction as, the net force acting on it. If the acceleration is zero, then the sum of the contributions from the Coriolis effect, the pressure gradient force, friction, and gravity is zero. We do not have to consider friction with the ground because we are at a high altitude, and we can neglect the force of gravity because we are concerned with horizontal motion. Therefore we are left with the equation

$$-2\vec{\Omega}\times\vec{U} - g\nabla h = \vec{0}$$

The wind $\vec{U}$ that satisfies this equation is called the geostrophic wind, and is denoted $\vec{V_g} = v_x\hat{\imath} + v_y\hat{\jmath}$.

(Caution: Do not think of v_x and v_y as partial derivatives; they are just the components of $\vec{V_g}$ in the northern and eastern directions.)

Substituting $\vec{V_g}$ for $\vec{U}$ we see that

$$\begin{aligned} -2\vec{\Omega}\times\vec{U} &= -2\vec{\Omega}\times\vec{V_g} \\ &= 2\Omega\sin(\phi)v_y\hat{\imath} + 2\Omega\sin(\phi)v_x\hat{\jmath} - 2\Omega\cos(\phi)v_x\hat{k}. \end{aligned}$$

< fix equation>

Again, we only care about the horizontal component of this vector, and

$$\begin{aligned} 2\Omega\sin(\phi)v_y\hat{\imath} - 2\Omega\sin(\phi)v_x\hat{\jmath} &= 2\Omega\sin(\phi)((v_x\hat{\imath} + v_y\hat{\jmath})\times\hat{k}) \\ &= 2\Omega\sin(\phi)(\vec{V_g}\times\hat{k}). \end{aligned}$$

Thus, to find an expression for the geostrophic wind, we must solve the equation

$$g\nabla h = 2\Omega\sin(\phi)(\vec{V_g}\times\hat{k}),$$

or equivalently

$$\frac{g}{2\Omega\sin(\phi)}\nabla h = \vec{V_g}\times\hat{k},$$

for $\vec{V_g}$. But how do we isolate an individual term of the cross product? Here we use an interesting trick. Notice that

$$\vec{V_g}\times\hat{k} = (v_x\hat{\imath} + v_y\hat{\jmath})\times\hat{k} = v_y\hat{\imath} - v_x\hat{\jmath}.$$

Take the cross product with $\hat{k}$ again to obtain

$$(\vec{V_g}\times\hat{k})\times\hat{k} = -v_x\hat{\imath} - v_y\hat{\jmath} = -\vec{V_g}.$$

Therefore,

$$\vec{V_g} = \frac{-g}{2\Omega\sin(\phi)}\left(\nabla h\times\hat{k}\right). \quad (1)$$

From this equality we conclude that the geostrophic wind blows perpendicular to the horizontal pressure gradient force, and the direction can be found using the right-hand rule. In particular, the upper-level winds blow parallel to the height contours on the pressure surface. So from a quick glance at the upper level charts we can tell which way the wind is blowing. Figure 5 shows the wind at 250 mb as a vector field as well as lines of constant wind speed for the same time as the 250 mb heights in figure 1. Where the lines of constant height are closer together, the pressure gradient force is large, causing the geostrophic wind to be stronger than in areas where the height contours are spaced farther apart. This is similar to the effect in which water in a river flows faster through narrow tunnels, or in which water sprays out of a hose when you place your thumb partially over the end.

[Figure 5 about here] Figure 5: The wind at 250 mb as a vector field and contours of constant wind speed on April 15, 2012.

Effects on Flight Times

Now that we understand the relationship between the upper level pressure surface and wind, we can compute approximate flight times.

Meteorologists can estimate the average geostrophic wind along a flight path from San Francisco to New York City by using data from the 250 mb pressure surface map. To do this, they overlay a path along a great circle from San Francisco to New York City and use equation (1) to estimate the wind at several points along this path. The spacing between the contour lines is a key component of the computation. In particular, the gradient of h is estimated using the distance between the contours, and the latitude ϕ is known for each point along the path. Then they use these values to find average wind along the flight path. This may feel familiar, as it is similar to how we estimate a line integral on a contour diagram in multivariable calculus.

Figure 6 shows the 30-year average heights for the 250 mb surface. Using this historical data, the geostrophic wind speed along the flight path is approximately 46 knots (nautical miles per hour). The total distance between the two airports is approximately 2250 nautical miles (nmi); only 2000 of those miles are spent at cruising altitude. The average cruising speed of most commercial aircraft is approximately 450 knots. Combining aircraft speed with the wind, the airplane will travel eastbound at 496 knots and westbound at 404 knots. This corresponds to roughly $\frac{2000 \text{ nmi}}{496 \text{ nmi/hr}} \cdot 60 \text{ min/hr} = 242$ minutes eastbound versus $\frac{2000 \text{ nmi}}{404 \text{ nmi/hr}} \cdot 60 \text{ min/hr} = 297$ minutes westbound. The difference in cruising time between these two directions is about 55 minutes—very close to what we found by examining the airline schedules!

[Figure 6 about here] Figure 6: Historical average for the 250 mb heights.

On any given day the wind pattern is not equal to the historical average. Two extreme examples are shown in figure 7. The top map is from July 3, 2002; there was very little wind over the continental United States that day, and the average wind speed along the flight path was a mere 17 knots. This means that the eastbound flights take 257 minutes, arriving 15 minutes later than the scheduled flight arrival time. The westbound flights take 277 minutes, arriving 20 minutes earlier than the estimated arrival time.

[Figure 7 about here] Figure 7: Two extreme examples of different wind patterns.
(a) 250 mb heights for July 3, 2002.
(b) 250 mb heights for Jan 18, 2012.

The bottom map in figure 7 is from January 18, 2012. Just looking at the tightly spaced contours, we see that the winds were fast. On that date the average wind speed was almost 90 knots along the flight path—enough to speed up the eastbound flight by 20 minutes and to slow down the westbound flight by 36 minutes.

These maps are available to the public and easy to access. Simply go to *http://weather.rap.ucar.edu/model/*. Then you can use them to determine if you will arrive early or late on your next cross-country flight. Bon voyage!

Further Reading

To learn more about the radiosonde balloon launches, read *http://www.nws.noaa.gov/om/wind/raob.shtml*.

For more information about wind, visit *http://www.nws.noaa.gov/om/wind/wnd.shtml.*

A couple good introductory meteorology texts are *The Atmosphere: An Introduction to Meteorology* by Frederick K. Lutgens, Edward J. Tarbuck and Dennis G. Tasa and *Understanding Weather and Climate* by Edward Aguado and James E. Burt.

Image Credits

Figures 1, 6, and 7 were made using the tools at the Earth System Research Laboratory, Physical Sciences Division, Daily Climate Composites. *http://www.esrl.noaa.gov/psd/data/composites/day/.*

Figure 2 is available at the Libre map project. *http://libremap.org.*

Figure 3 is from
The Radiosonde Musuem of North America, http://radiosondemuseum.org/what-is-a-radiosonde/ (if you use just the balloon) or National Weather Service-Rapid City, SD http://www.crh.noaa.gov/unr/?n=upper-air (if you use the pic with the person and balloon and building)

Figure 5 was made using the tools available at the National Oceanic and Atmospheric Administration, CDAS-NCEP/NCAR Reanalysis, *http://nomad3.ncep.noaa.gov/cgi-bin/pdisp_6p_r1.sh.*

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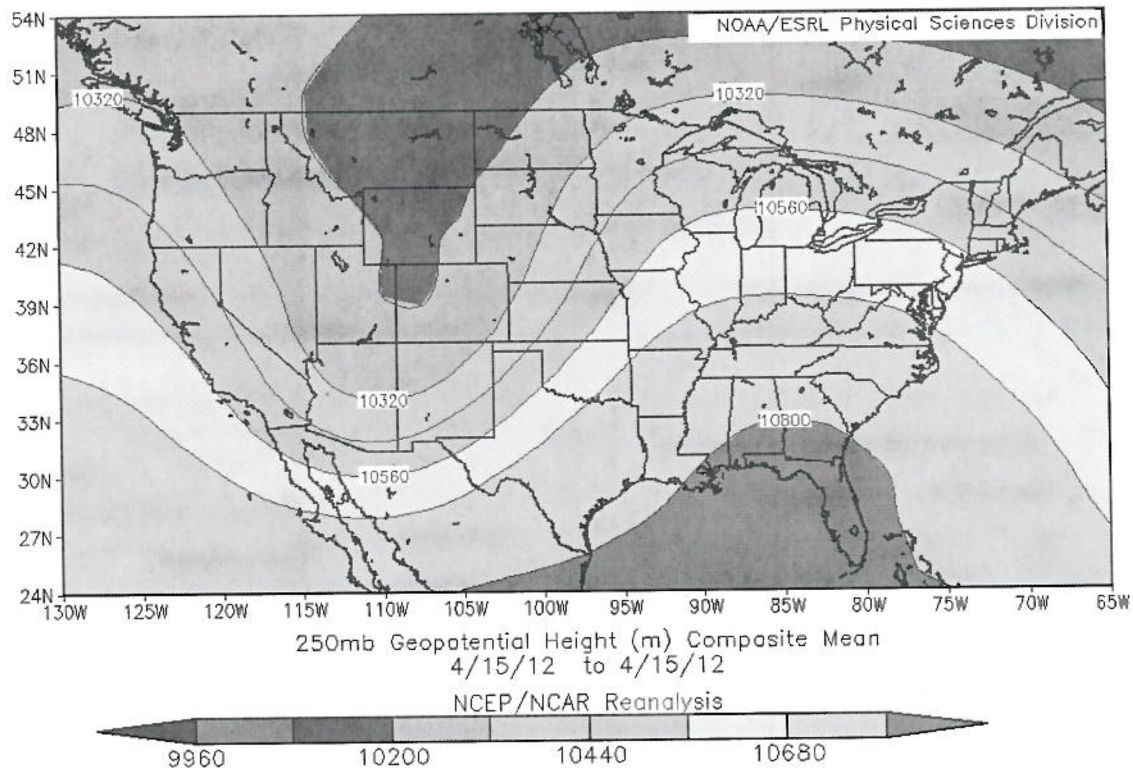


250mb Geopotential Height (m) Composite Mean
4/15/12 to 4/15/12
NCEP/NCAR Reanalysis

Fig 1

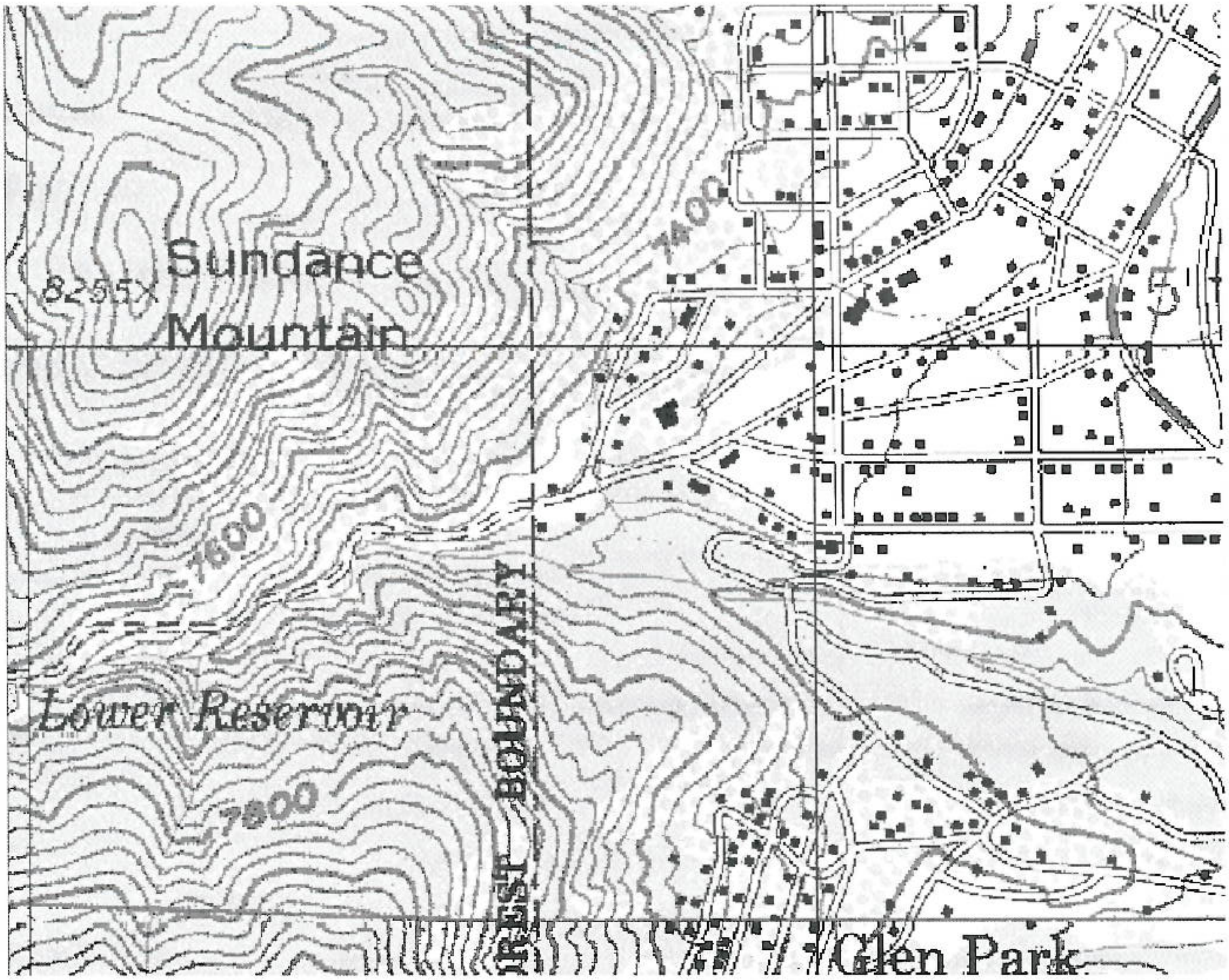

Sundance
Mountain
8255X
7400
7600
7800
Lower Reservoir
FOREST BOUNDARY
Glen Park

Fig2

Fig 3 OR

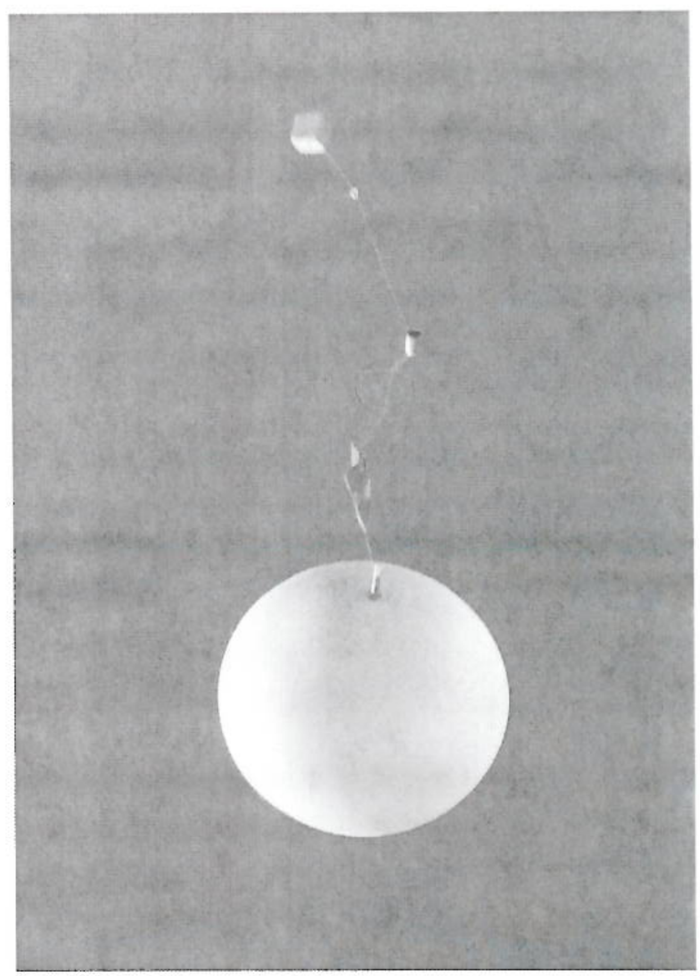


Fig 3

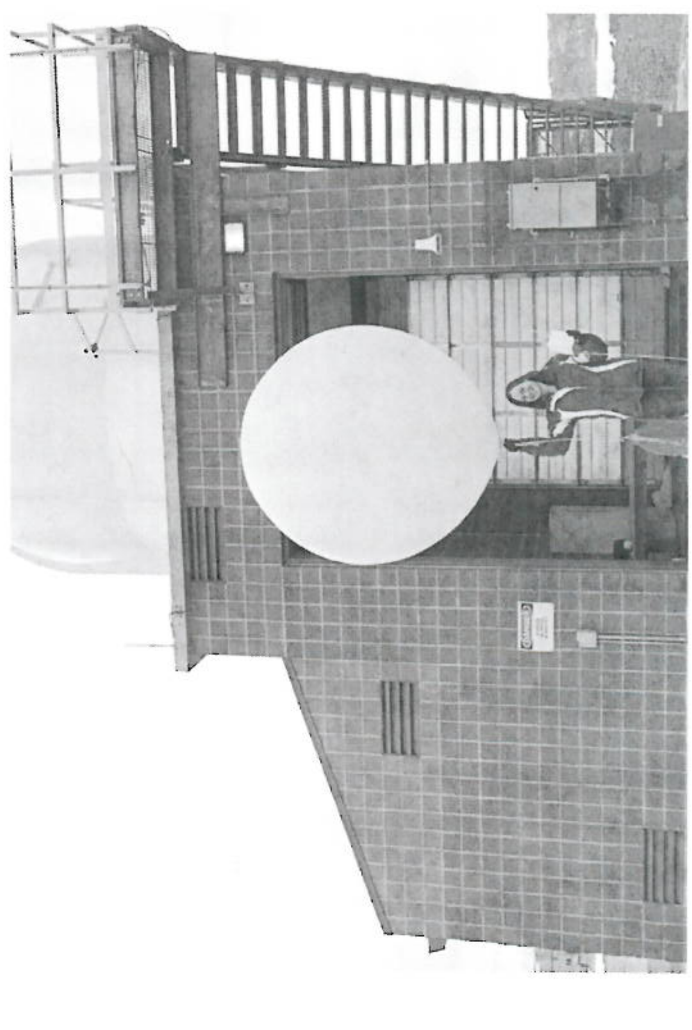
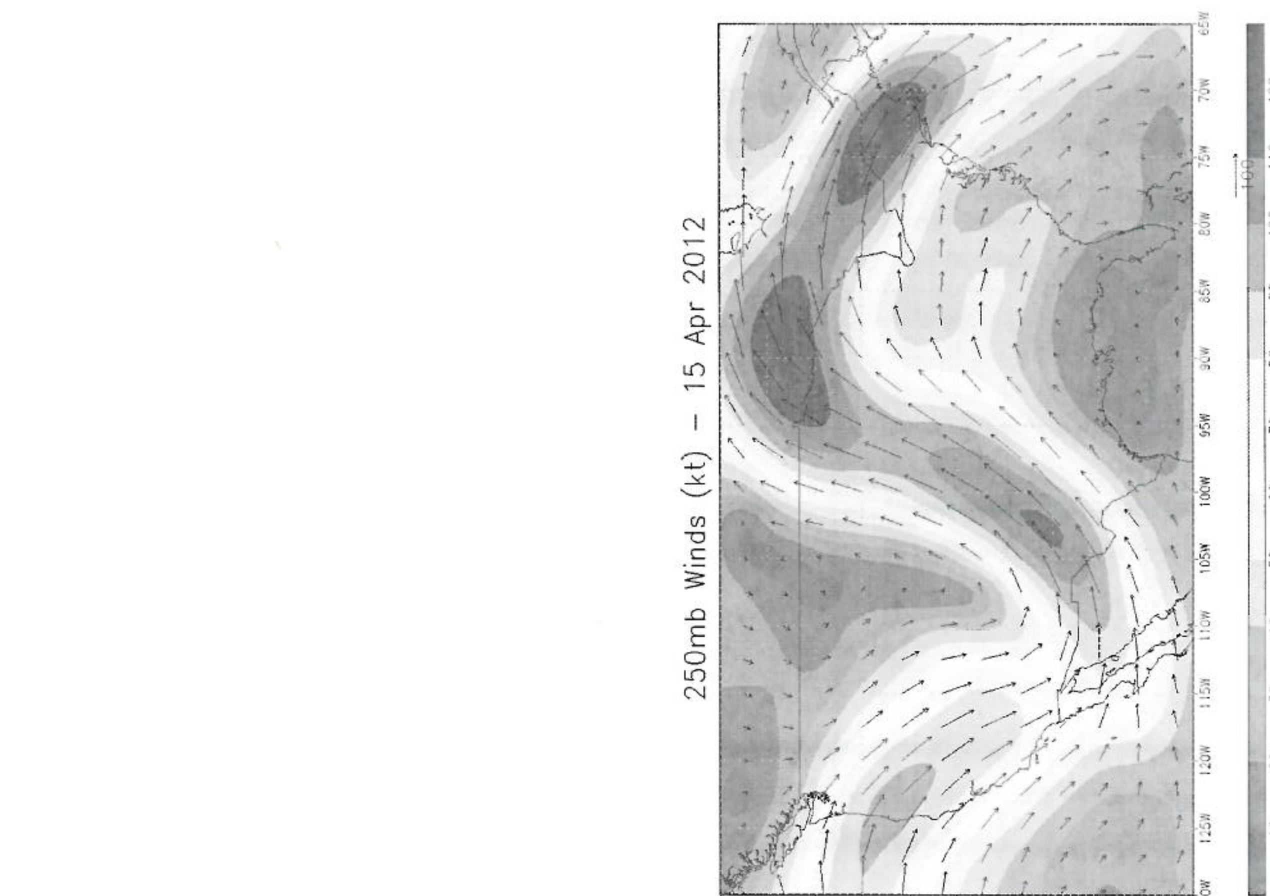


Fig 4



250mb Winds (kt) - 15 Apr 2012
54N
51N
48N
45N
42N
39N
36N
33N
30N
27N
24N
130W
125W
120W
115W
110W
105W
100W
95W
90W
85W
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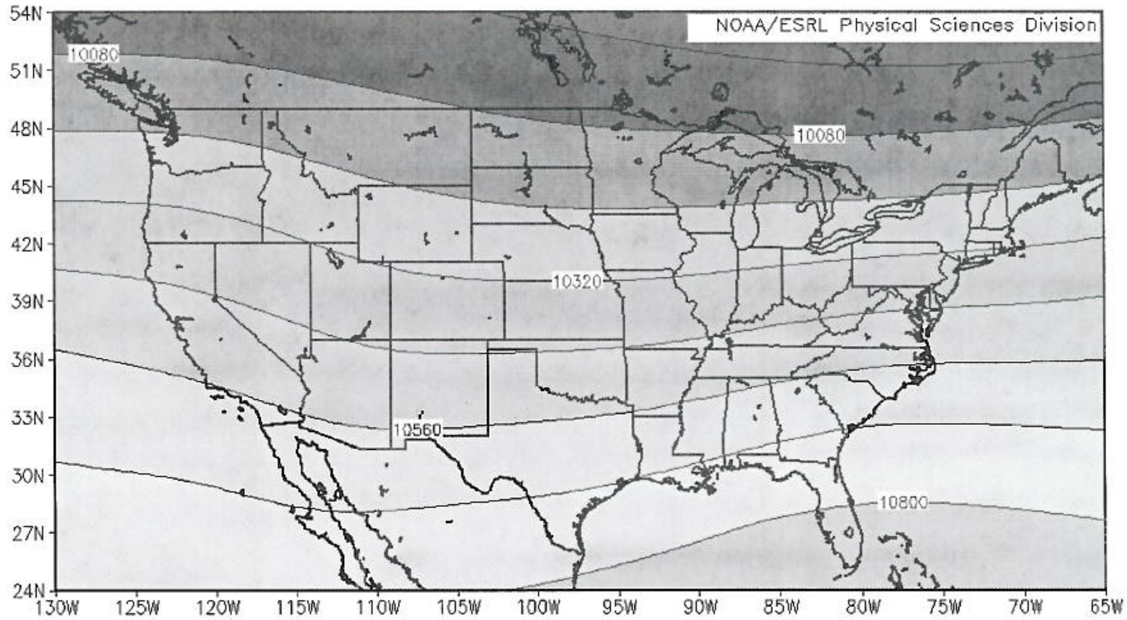
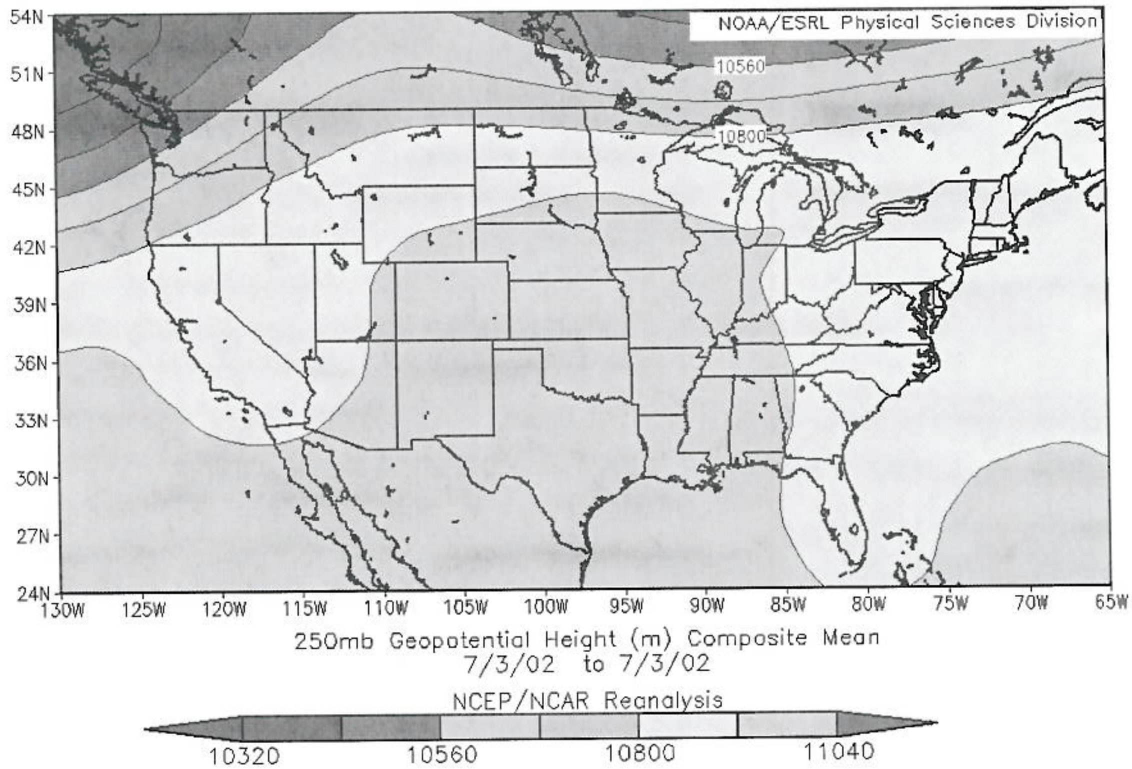
90
100
110
120
100



250mb Geopotential Height (m) Climatology (1981–2010 Climatology)
12/14

NCEP/NCAR Reanalysis

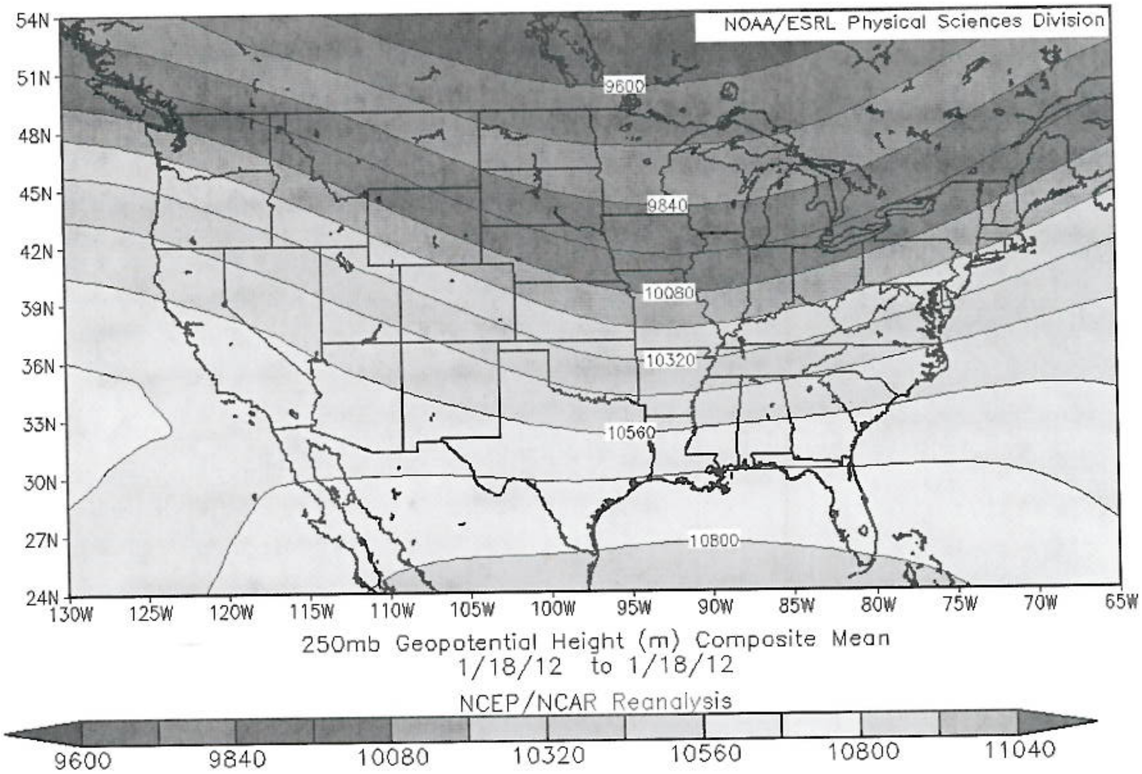
Fig 6



250mb Geopotential Height (m) Composite Mean
7/3/02 to 7/3/02

NCEP/NCAR Reanalysis

Fig 7a



250mb Geopotential Height (m) Composite Mean
1/18/12 to 1/18/12
NCEP/NCAR Reanalysis

Fig 7b